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Adiabatic elimination for classical fermionic systems

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Abstract. We present the reduction to a normal form of a Grassmannian differential equation describing a classical fermionic system in the neighbourhood of an instability. As an application of the method presented we show that the anticommuting version of the normal form associated with (a) the Hopf bifurcation (a simple pair of imaginary eigenvalues) leads to the overdamped Grassmann harmonic oscillator, and (b) the resonant Hopf bifurcation 1 : 1 (a double pair of semisimple imaginary eigenvalues) corresponds exactly to the Thirring model for Grassmann solitons.

1. Introduction

A great deal of attention has been devoted recently to the study of Grassmann variables in connection with the description of Fermi systems (Berezin and Marinov 1977, Elphick 1986, Ohnuki and Kamefuchi 1980). This stems from the fact that Grassmann variables are the classical counterparts of fermionic quantum operators or, in other words, for each Fermi operator one can construct an eigenbasis of coherent states such that its eigenvalues belong to a Grassmann algebra. Therefore the classical version of the Heisenberg equations of motion for a Fermi system corresponds to an evolution equation whose phase space is a Grassmann algebra, i.e. a differential equation for anticommuting variables. Particular models involving this type of equation have been extensively studied by Berezin and Marinov (1977) in the context of supersymmetry, by Kulish and Nissimov (1976) who considered the anticommuting massive Thirring model (AMTM) and by Morris (1978) who determined a Bäcklund transformation for the AMTM generalising the prolongation structure method of Wahlquist and Estabrook (1975) to Grassmann algebra-valued differential forms.

It is worthwhile pointing out that differential equations for Grassmann variables are also naturally encountered in the description via the path integral of a quantum mechanical Fermi system. In this description, if $H(a_j, a_j^\dagger, t)$ denotes the normal ordered Hamiltonian of the fermionic system then the integral kernel of the evolution operator $U(\theta_{(f)}, t_f; \theta_{(i)}, t_i)$ ($\theta_{(f)}, \theta_{(i)}$ are Grassmann variables and * is the analogue of †) is given by a functional integral over a Grassmann algebra:

$$U(\theta_{(f)}, t_f; \theta_{(i)}, t_i) = \int \mathscr{D}(\theta_j^*, \theta_j) \exp\left[\tfrac{1}{2}(\theta^*_{(f)k}\theta_{(f)k} + \theta^*_{(i)k}\theta_{(i)k}) + i\int_{t_i}^{t_f} dt \left(\frac{1}{2i}(\theta_k^*\dot{\theta}_k - \dot{\theta}_k^*\theta_k) - h(\theta_j, \theta_j^*, t)\right)\right] \tag{1}$$

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where h is the 'classical value' of H when we replace the operators $a_j^\dagger, a_j$ by the Grassmann variables θ_j^*, θ_j. Explicit calculation of the path integral leads to the evaluation of the classical action on the extremal trajectory satisfying the classical equations of motion:

$$\frac{1}{\mathrm{i}}\dot{\theta}_j=\frac{\partial h}{\partial\theta_j^*} \qquad \frac{1}{\mathrm{i}}\dot{\theta}_j^*=\frac{\partial h}{\partial\theta_j} \tag{2}$$

(since θ_j, θ_j^* anticommute the sign in the Hamilton equations is the same). Therefore we have naturally arrived at a set of Grassmannian differential equations. (If a fermionic field theory is considered then (2) is a set of partial differential equations (PDE) for Grassmann fields (this type of equation has recently received much attention in connection with supersymmetric extensions of integrable models and superevaluation equations (Gürses and Oğuz 1985, Kupershmidt 1984, Olshanetsky 1983, Roy Chowdhury and Roy 1986)). For the sake of clarity we consider here only the case of a finite number of degrees of freedom, the generalisation to PDE for Grassmann fields being straightforward.)

We consider here, with no particular model in mind, a general non-linear Grassmann differential equation (not necessarily derived from a variational principle) describing a classical Fermi system undergoing a degenerate bifurcation. Such a situation is characterised by the linear instability of one or several Grassmann modes (critical modes) while the others remain strongly damped. We show that by an appropriate non-linear change of variables the original differential system can be reduced to a simpler one such that its asymptotic dynamics is described only by an equation for the critical modes referred to as the normal form equation.

The plan of this paper is as follows: in § 2 we present some preliminaries on Grassmann algebras and notation to be used in the other sections. Section 3 is devoted to developing the method leading to the normal form and the equation for the non-critical modes. In § 4 we make a useful comment on adiabatic elimination for Grassmann variables and finally in § 5 we apply the techniques presented to some specific examples of low codimension (minimum number of parameters needed to unfold the critical situation) and, in particular, we study how the normal form corresponding to the double Hopf bifurcation (two pairs of semisimple imaginary eigenvalues) leads to the Thirring model.

2. Brief preliminary on Grassmann algebras

An algebra whose generators $\xi_1, \ldots, \xi_n$ satisfy the relations

$$[\xi_i, \xi_j]_- = \xi_i\xi_j + \xi_j\xi_i = 0 \qquad i, j = 1, \ldots, n \tag{3}$$

is called a Grassmann algebra $\mathscr{G}_n$ with n generators. If an algebra $\mathscr{G}$ possesses an infinite countable number of generators satisfying (3) for any i, j belonging to a countable set I then we will refer to $\mathscr{G}$ as an infinite-dimensional Grassmann algebra. $\mathscr{G}_n$ is also a 2^n-dimensional linear space which is the direct sum of linear spaces

$$\mathscr{G}_n = \bigoplus_k \mathscr{G}_n^{(k)} \tag{4}$$

where $\mathscr{G}_n^{(k)}$ is $\binom{n}{k}$ dimensional and generated by monomials of degree k $\{\xi_{i_1}\xi_{i_2}\ldots\xi_{i_k}\}$. $\mathscr{G}_n^{(0)}$ is a one-dimensional linear space whose generator (the identity of $\mathscr{G}_n$) can be regarded as the generator of $\mathbb{C}$.

It follows that an arbitrary element g of $\mathscr{G}$ can be represented in the form of a linear combination of monomials

$$g=\alpha_0+\sum_{i\in I}\alpha_i\xi_i+\sum_{i_1,i_2\in I}\alpha_{i_1i_2}\xi_{i_1}\xi_{i_2}+\ldots+\sum_{i_1,i_2,\ldots i_j\in I}\alpha_{i_1i_2\ldots i_j}\xi_{i_1}\xi_{i_2}\ldots\xi_{i_j}+\ldots. \tag{5}$$

This decomposition is unique if all the coefficients $\alpha_{i_1i_2\ldots i_j}\in\mathbb{C}$ are totally antisymmetric in $i_1,\ldots,i_j\in I$. An element in $\mathscr{G}$ of the form

$$\sum_{i_1,i_2,\ldots,i_k}\alpha_{i_1i_2\ldots i_k}\xi_{i_1}\xi_{i_2}\ldots\xi_{i_k} \tag{6}$$

is called a homogeneous element of degree k. Therefore $\mathscr{G}$ can be decomposed as $\mathscr{G}_e\oplus\mathscr{G}_o$ where $\mathscr{G}_e$ (resp $\mathscr{G}_o$) is the set of elements in $\mathscr{G}$ which are linear combinations of homogeneous elements of even (resp odd) degree. Elements belonging to $\mathscr{G}_e$ (resp $\mathscr{G}_o$) are called even (resp odd). It follows that for any g in $\mathscr{G}$, $g\mathscr{G}_e=\mathscr{G}_eg$ and $g\mathscr{G}_o=-\mathscr{G}_og$ (for g odd).

If $\mathscr{G}$ is endowed with an inner product $\langle\ ,\ \rangle$ then we can define a one-to-one mapping of $\mathscr{G}$ onto itself $g\to g^*$ such that (i) $(g^*)^*=g$, (ii) $(g_1g_2)^*=g_2^*g_1^*$, (iii) $(\alpha g)^*=\bar{\alpha}g^*$, $\alpha\in\mathbb{C}$, and (iv) $\langle f,g\rangle=\langle f^*,g^*\rangle$. This mapping is called the involution in $\mathscr{G}$ and the elements g, g^* are called adjoint to each other.

Finally we review some basic properties of derivation and integration of Grassmann algebras. For the sake of clarity we will illustrate these concepts with the following simple example. We consider a two-level Fermi system with the two operators a, a^+ satisfying $aa^++a^+a=1$, $a^2=(a^+)^2=0$. We represent them acting on the four-dimensional Grassmann algebra generated by θ, θ^* ($\theta^2=(\theta^*)^2=0$, $\theta\theta^*+\theta^*\theta=0$). An element g in $\mathscr{G}_2$ has the form

$$g(\theta,\theta^*)=g_0+g_1\theta+g_2\theta^*+g_3\theta\theta^*. \tag{7}$$

We introduce in $\mathscr{G}_2$ the linear left derivation

$$\partial^lg/\partial\theta=g_1+g_3\theta^* \qquad \partial^lg/\partial\theta^*=g_2-g_3\theta. \tag{8}$$

Decomposing $\mathscr{G}_2$ as $\mathscr{G}_{2(e)}\oplus\mathscr{G}_{2(o)}$ we easily derive the following properties. Let g be an arbitrary element in $\mathscr{G}_2$; then

$$\frac{\partial^l(g_1g)}{\partial\xi}=\left(\frac{\partial^lg_1}{\partial\xi}\right)g\pm g_1\frac{\partial^lg}{\partial\xi} \tag{9}$$

where the + sign (resp −) holds if $g_1\in\mathscr{G}_{2(e)}$ (resp $\mathscr{G}_{2(o)}$) and ξ stands for θ or θ^*.

If $g,f\in\mathscr{G}_2$ then we define the scalar product by

$$\langle g,f\rangle=g_0\bar{f}_0+g_1\bar{f}_1+g_2\bar{f}_2+g_3\bar{f}_3 \tag{10}$$

(a bar stands for complex conjugate) which can be written as the Grassmann integral

$$\int\exp(-(\theta_1^*\theta_1+\theta_2^*\theta_2))g(\theta_1,\theta_2)\bar{f}(\theta_2^*,\theta_1^*)\,\mathrm{d}\theta_1^*\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2^*\,\mathrm{d}\theta_2 \tag{11}$$

where the integral symbol is defined by linearity starting from the requirements (Berezin 1966)

$$\int\mathrm{d}\theta=\int\mathrm{d}\theta^*=0 \qquad \int\theta\,\mathrm{d}\theta=\int\theta^*\,\mathrm{d}\theta^*=1 \qquad [\mathrm{d}\theta,\mathrm{d}\theta^*]_-=0 \tag{12}$$

and (θ,θ^*) anticommute with $(\mathrm{d}\theta,\mathrm{d}\theta^*)$. From (8) and (12) we clearly see that the integral and the left derivation are identical.

The above properties are easily generalised to several degrees of freedom.

3. Reduction to the normal form and the equation for the stable modes

Let $\mathscr{G}$ be an infinite-dimensional complex Grassmann algebra with an involution. Let us consider in $\mathscr{G}$ a finite family $\mathscr{F}$ of n real odd elements depending on a real parameter t:

$$\mathscr{F} = \{\theta_i(t) \in \mathscr{G}_o;\ \theta_i(t) = \theta_i^*(t), [\theta_i(t), \theta_j(t)] = 0,\ \forall i, j = 1, \ldots, n,\ \forall t \in [t_0, +\infty)\}. \tag{13}$$

An element θ_i in $\mathscr{F}$ can be represented as

$$\theta_i(t) = \sum_{j \in I} f_i^j(t)\xi_j + \sum_{j_1 j_2 j_3 \in I} f_i^{j_1 j_2 j_3}(t)\xi_{j_1}\xi_{j_2}\xi_{j_3} + \ldots \tag{14}$$

where the coefficients $f_i^{j_1 j_2 \ldots j_p}(t)$ belong to $C^\infty([t_0, \infty))$.

We assume that elements in $\mathscr{F}$ describe a classical Fermi system and evolve with t according to

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} = L\boldsymbol{\theta} + \boldsymbol{N}(\boldsymbol{\theta}) \qquad \boldsymbol{\theta}(t = t_0) = \boldsymbol{\theta}_0 \tag{15}$$

where $\boldsymbol{\theta}$ is a n-dimensional Grassmann vector, $\boldsymbol{\theta} = \theta_i \boldsymbol{e}^i$ (sum over repeated indices), $\boldsymbol{e}^i$, $i = 1, \ldots, n$, being the canonical basis of a vectorial space E, L is a real linear operator acting on E and $\boldsymbol{N}(\boldsymbol{\theta})$ stands for an arbitrary odd non-linearity (note that $\mathrm{d}\boldsymbol{\theta}/\mathrm{d}t$ is odd and if (15) is a Hamiltonian system, as in (2), then h must belong to $\mathscr{G}_e$). Note that if $n = 2$ then (15) reduces to a linear equation. In the following, $n > 2$ will be implicitly assumed. We also suppose that L and $\boldsymbol{N}$ depend on a parameter $\mu \in \mathbb{R}^P$ and that in (15) we are at a point μ_c such that L has n_c eigenvalues with a vanishing real part which we call critical, while the rest of the eigenvalues denoted by γ_α, $\alpha = 1, \ldots, M = n - n_c$, are different and have a strictly negative real part. In this scenario, assumed to persist in a small neighbourhood of μ_c, the asymptotic solution of (15) $\boldsymbol{\theta} = 0$ is linearly unstable. Since L is real, the critical eigenvalues can only be zero or pure imaginary, in which case they are complex conjugate $(\mathrm{i}\omega_k, -\mathrm{i}\omega_k)$ $k = 1, \ldots, s$ (for simplicity we assume they are semisimple). Then $n_c = l' + 2s$ where l' is the algebraic multiplicity of the zero eigenvalue.

According to our assumptions we can split E into the direct sum of L-invariant subspaces $E_c \oplus E_s$, where E_s is the stable subspace spanned by the vectors $\boldsymbol{\psi}_\alpha$, $L\boldsymbol{\psi}_\alpha = \gamma_\alpha \boldsymbol{\psi}_\alpha$, $\alpha = 1, \ldots, M$, and E_c is spanned by the generalised eigenvectors $\boldsymbol{\phi}_i$, $i = 1, \ldots, n_c$, such that

$$L\boldsymbol{\phi}_i = \mathbb{J}_{ji}\boldsymbol{\phi}_j \tag{16}$$

where the Jordan matrix is given by

$$\mathbb{J} = \begin{bmatrix} \mathbb{J}_1 & & & & & & \\ & \mathbb{J}_2 & & & & & \\ & & \ddots & & & & \\ & & & \mathbb{J}_q & & & \\ & & & & 0 & & \\ & & & & & \Omega_1 & \\ & & & & & & \ddots \\ & & & & & & & \Omega_s \end{bmatrix} \tag{17}$$

where $\mathbb{J}_i$ is a $n_i \times n_i$ matrix with zeros in the diagonal and ones in the upper diagonal, 0 is a $l \times l$ null matrix, Ω_j is the diagonal matrix corresponding to the pair $(\mathrm{i}\omega_j, -\mathrm{i}\omega_j)$ and $\Sigma_{i=1}^q n_i + l = l'$.

Let us consider now in $\mathscr{G}$ a new finite family of odd variables

$\{A_1, A_2, \ldots, A_{l'}, A_{l'+1}, A_{l'+2} = A^*_{l'+1}, \ldots, A_{n_c-1}, A_{n_c} = A^*_{n_c-1}; B_\alpha, \alpha = 1, \ldots, M\}$

defined through the non-linear change of variables

$$\boldsymbol{\theta}(t) = \sum_{i=1}^{n_c} A_i(t)\boldsymbol{\phi}_i + \sum_{\alpha=1}^{M} B_\alpha(t)\boldsymbol{\psi}_\alpha + \sum_{j=1}^{(m-1)/2} \boldsymbol{\theta}^{[2j+1]}(A_i, B_\alpha) \tag{18}$$

where $m = n$ (resp $m = n-1$) if n is odd (resp even) and $\boldsymbol{\theta}^{[2j+1]}$ is homogeneous of degree $2j+1$ in the variables $\{A_i, B_\alpha\}$. We look for equations for $\{A_i, B_\alpha\}$ of the form

$$\frac{dA_i}{dt} = \mathbb{J}_{ij}A_j + \sum_{j=1}^{(m-1)/2} f_i^{[2j+1]}(A_k, B_\beta) \equiv \mathbb{J}_{ij}A_j + F_i \qquad i = 1, \ldots, n_c \tag{19a}$$

$$\frac{dB_\alpha}{dt} = \gamma_\alpha B_\alpha + \sum_{j=1}^{(m-1)/2} g_\alpha^{[2j+1]}(A_k, B_\beta) \equiv \gamma_\alpha B_\alpha + G_\alpha \qquad \alpha = 1, \ldots, M \tag{19b}$$

where $f_i^{[2j+1]}$, $i = 1, \ldots, n_c$, and $g_\alpha^{[2j+1]}$, $\alpha = 1, \ldots, M$ are homogenous of degree $2j+1$ in $\{A_k, B_\beta\}$. The idea is to choose $\boldsymbol{\theta}^{[2j+1]}, j = 1, \ldots, \frac{1}{2}(m-1)$, such that F_i, G_α can be taken as simple as possible. As we will see below, G_α will turn out to be linear in B_α, F_i independent of B_β, $\beta = 1, \ldots, M$, and such that $\boldsymbol{F} = F_i\boldsymbol{\phi}_i$ is equivariant under the one-parameter Lie group generated by $\mathbb{J}^\dagger$ (Elphick *et al* 1986).

From (18) we have

$$\frac{d\boldsymbol{\theta}}{dt} = \frac{dA_i}{dt}\frac{\partial^l\boldsymbol{\theta}}{\partial A_i} + \frac{dB_\alpha}{dt}\frac{\partial^l\boldsymbol{\theta}}{\partial B_\alpha} \tag{20}$$

where the subscript l means that in differentiating with respect to the variable A_i or B_α we must displace it to the left before dropping it. By substituting (18) into (15), using (20) and equations (19*a*) and (19*b*) we obtain the following hierarchy of equations ($j = 0$ leads to a trivial verification):

$$\begin{aligned}\mathscr{L}\boldsymbol{\theta}^{[2j+1]} &= (\mathscr{A} + \mathscr{B} - L)\boldsymbol{\theta}^{[2j+1]} \\ &= \boldsymbol{I}^{[2j+1]} - f_i^{[2j+1]}\boldsymbol{\phi}_i - g_\alpha^{[2j+1]}\boldsymbol{\psi}_\alpha \equiv \boldsymbol{K}^{[2j+1]} \qquad j = 1, \tfrac{1}{2}(m-1)\end{aligned} \tag{21}$$

where

$$\mathscr{A} = \mathbb{J}_{ij}A_j\, \partial^l/\partial A_i \qquad \mathscr{B} = \gamma_\alpha B_\alpha\, \partial^l/\partial B_\alpha \tag{22}$$

and

$$\boldsymbol{I}^{[2j+1]} = \boldsymbol{N}^{[2j+1]} - \sum_{k=1}^{j-1}\left[\left(\frac{dA_i}{dt}\right)^{[2k+1]}\frac{\partial^l}{\partial A_i} + \left(\frac{dB_\alpha}{dt}\right)^{[2k+1]}\frac{\partial^l}{\partial B_\alpha}\right]\boldsymbol{\theta}^{[2(j-k)+1]}. \tag{23}$$

If $j = 1$, the last term in (23) is absent. The operator $\mathscr{L} = \mathscr{A} + \mathscr{B} - L$ in (21) is called the homological operator associated to L (Arnold 1977, Elphick *et al* 1986) and acts on the tensor product $\mathscr{H} = E \otimes \mathscr{H}_1 \otimes \mathscr{H}_2$ where $\mathscr{H}_1$ (resp $\mathscr{H}_2$) is the linear space generated by odd monomials in the variables A_i (resp B_α). We endow $\mathscr{H}$ with the inner product

$$\langle\ ,\ \rangle_{\mathscr{H}} = \langle\ ,\ \rangle_E \cdot \langle\ ,\ \rangle_{\mathscr{H}_1} \cdot \langle\ ,\ \rangle_{\mathscr{H}_2} \tag{24}$$

where $\langle\ ,\ \rangle_E$ is such that $\{\boldsymbol{\phi}_i, \boldsymbol{\psi}_\alpha\}$ form an orthonormal basis and $\langle\ ,\ \rangle_{\mathscr{H}_1}$ is given by (an analogous expression holds for $\mathscr{H}_2$)

$$\langle f_1, f_2\rangle_{\mathscr{H}_1} = \int \exp\left(-\sum_{i=1}^{n_c} A_i^* A_i\right) f_1(A_j) f_2^*(A_j)\, dA_{n_c}^*\, dA_{n_c} \ldots dA_1^*\, dA_1 \qquad \forall f_1, f_2 \in \mathscr{H}_1. \tag{25}$$

We note that the above expression is a multiple Grassmann integral defined in the sense of § 2. It is worthwhile remarking that the scalar product introduced for $\mathscr{H}_1$ (resp $\mathscr{H}_2$) is such that the operators A_i, $\partial^l/\partial A_i$ (resp B_α, $\partial^l/\partial B_\alpha$) are adjoint to each other and satisfy

$$\begin{gathered}(a_i^{\dagger}=A_i,\ a_i=\partial^l/\partial A_i,\ b_\alpha^{\dagger}=B_\alpha,\ b_\alpha=\partial^l/\partial B_\alpha)\\ [a_i^{\dagger},a_i]_-=\delta_{ij}\qquad [b_\alpha^{\dagger},b_\beta]_-=\delta_{\alpha\beta}\end{gathered}\tag{26}$$

i.e. they are fermionic creation and annihilation operators in the Fock space $\mathscr{H}_1$ (resp $\mathscr{H}_2$) or, equivalently, $a_j+a_j^{\dagger}$, $\mathrm{i}(a_j-a_j^{\dagger})$ (resp $b_\alpha+b_\alpha^{\dagger}$, $\mathrm{i}(b_\alpha-b_\alpha^{\dagger})$) generate the Clifford algebra C_{2n_c} (resp C_{2M}).

Using the above properties of the inner product in $\mathscr{H}$, the adjoint of $\mathscr{L}$ becomes

$$\mathscr{L}^{\dagger}=\bar{\mathbb{J}}_{ij}A_i\,\partial^l/\partial A_j+\bar{\gamma}_\alpha B_\alpha\,\partial^l/\partial B_\alpha-L^{\dagger}\qquad L^{\dagger}={}^{\mathrm{t}}\bar{L}.\tag{27}$$

Therefore the adjoint of $\mathscr{L}$ corresponds to the homological operator associated to $L^{\dagger}$. Graphically, this means that the following diagram is commutative:

$$\begin{array}{ccc} L & \longrightarrow & \mathscr{L}\\ {\scriptstyle\text{adjoint in } E}\big\downarrow & & \big\downarrow{\scriptstyle\text{adjoint in } \mathscr{H}}\\ L^{\dagger} & \longrightarrow & \mathscr{L}^{\dagger}\end{array}.\tag{28}$$

Having defined $\mathscr{L}^{\dagger}$ we now focus our attention on equation (21). Since $\ker L$ is non-trivial it easily follows that $\mathscr{L}$ is a linear non-invertible operator in $\mathscr{H}$. Therefore (21) will not have solutions unless $\boldsymbol{K}^{[2j+1]}\in\operatorname{ran}\mathscr{L}$, $j=1,\ldots,\frac{1}{2}(m-1)$, from which we deduce that the unknown functions $f_i^{[2j+1]}\boldsymbol{\phi}_i+g^{[2j+1]}\boldsymbol{\psi}_\alpha$, $j=1,\ldots,\frac{1}{2}(m-1)$, have to be taken in the complementary space $\ker\mathscr{L}^{\dagger}$ ($\mathscr{H}=\operatorname{ran}\mathscr{L}\oplus\ker\mathscr{L}^{\dagger}$) modulo (if needed to simplify the form of (19a) and (19b)) some judicious choice in $\operatorname{ran}\mathscr{L}$. This solvability condition is nothing but the Fredholm alternative. Therefore we have to carry out the following programme: (a) determine the generators of $\ker\mathscr{L}^{\dagger}$ and (b) impose the Fredholm alternative $\boldsymbol{K}^{[2j+1]}$ orthogonal to $\ker\mathscr{L}^{\dagger}$ to determine $f_i^{[2j+1]}$, $g_\alpha^{[2j+1]}$.

Assuming the non-resonant condition of the set $\{\gamma_1,\ldots,\gamma_M,\gamma_{M+1}=\mathrm{i}\omega_1,\ldots,\gamma_{M+2s}=-\mathrm{i}\omega_s\}$

$$\gamma_i\neq\sum_{j\in J}\gamma_j\qquad\forall i\in I=\{1,\ldots,M+2s\},\ \forall J\subseteq I\tag{29}$$

we easily obtain that $\ker\mathscr{L}^{\dagger}$ is generated by vectors in $\mathscr{H}$ of the form

$$B_\alpha Y(A_j)\boldsymbol{\psi}_\alpha\qquad(\text{no sum over }\alpha)\tag{30a}$$

where $Y\in\ker\mathscr{A}^{\dagger}$ (Y must be an even element in $\mathscr{H}_1$), and by vectors $\boldsymbol{X}$ in $\mathscr{H}$ such that

$$(\mathscr{A}^{\dagger}-L^{\dagger})\boldsymbol{X}=0.\tag{30b}$$

From (30b) it follows that $\boldsymbol{X}$ is of the form

$$\sum_{i=1}^{n_c}X_i(A_j)\boldsymbol{\phi}_i\tag{31}$$

where $X_i(A_j)$, $i=1,\ldots,n_c$, are linear combinations of monomials of odd degree in $\mathscr{H}_1$.

We conclude that the functions F_i in (19a) can be chosen independent of $\{B_1,\ldots,B_M\}$ and such that $\boldsymbol{F}=F_i\boldsymbol{\phi}_i$ satisfies the differential equation

$$(\bar{\mathbb{J}}_{ij}A_i\,\partial^l/\partial A_j-\mathbb{J}^{\dagger})\boldsymbol{F}=0\tag{32}$$

which can be rewritten in the following equivalent form:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(\mathrm{e}^{-\mathbb{J}^{\dagger}\eta}\boldsymbol{F}(\mathrm{e}^{\mathbb{J}^{\dagger}\eta}\boldsymbol{A}))=0 \qquad \forall \boldsymbol{A}=\sum_{i=1}^{n_c} A_i\boldsymbol{\phi}_i \in E_c \tag{33}$$

and therefore $\boldsymbol{F}$ is equivariant under $\mathrm{e}^{\mathbb{J}^{\dagger}\eta}$ (η is a real parameter).

From (30*a*), (30*b*) and (31), and using $\langle\boldsymbol{\phi}_i, \boldsymbol{\psi}_\alpha\rangle_E=0$, $\forall i, \alpha$, we deduce that the functions G_α in (19*b*) are linear in B_α:

$$G_\alpha = B_\alpha H_\alpha(A_j) \qquad \alpha=1,\dots,M \tag{34}$$

where the functions H_α satisfy

$$\bar{\mathbb{J}}_{ij}A_i\frac{\partial^l}{\partial A_j}H_\alpha=0. \tag{35}$$

Using (35) we obtain that G_α satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{e}^{-\bar{\gamma}_\alpha\eta}(\mathrm{e}^{\gamma_\alpha\eta}B_\alpha H_\alpha(\mathrm{e}^{\mathbb{J}^{\dagger}\eta}\boldsymbol{a}))=0 \qquad \forall\alpha=1,\dots,M \tag{36}$$

which implies that $\boldsymbol{G}=\Sigma_{\alpha=1}^{M} G_\alpha\boldsymbol{\psi}_\alpha$ verifies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{e}^{-\Lambda^{*}\eta}G\left(\mathrm{e}^{L^{\dagger}\eta}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\right)=0 \qquad \forall\boldsymbol{A}\in E_c, \boldsymbol{B}\in E_s \tag{37}$$

where $\Lambda=\mathrm{diag}(\gamma_\alpha, \alpha=1,\dots,M)$.

Therefore, we conclude that the normal form equations (normal form for the critical modes plus the equation for the stable modes) (19*a*) and (19*b*):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}=L\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}+\begin{pmatrix}\boldsymbol{F}\\ \boldsymbol{G}\end{pmatrix} \tag{38}$$

are characterised by the equivariance property ($R_\eta=\mathrm{e}^{L^{\dagger}\eta}$):

$$R_\eta^{-1}\begin{pmatrix}\boldsymbol{F}(R_\eta\boldsymbol{C})\\ \boldsymbol{G}(R_\eta\boldsymbol{C})\end{pmatrix}=\begin{pmatrix}\boldsymbol{F}(\boldsymbol{C})\\ \boldsymbol{G}(\boldsymbol{C})\end{pmatrix} \qquad \forall\boldsymbol{C}=\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\in E \tag{39}$$

which follows directly from (33) and (37), and leads to the simplest form for $\boldsymbol{F}$ and $\boldsymbol{G}$. We note that although we have used (29) to derive (39) a much more involved calculation also leads to (39) even in the case where (29) does not hold and L admits eigenvalues with a positive real part.

It is worth remarking that (39) is a global characterisation for the normal form (38) giving *a priori* the simplest functional form for the non-linearities in (38). To know explicitly the coefficients of each admissible monomial (odd) we have to use the Fredholm alternative in (21).

We finally mention that in order to consider the unfolded situation in (15), i.e. when $\mu=\mu_c+\delta\mu$, it suffices to replace the critical part of L in (38) by the corresponding Arnold–Jordan matrix (Arnold 1971, Elphick *et al* 1986).

4. Remarks

It is worthwhile noting that since the functions G_α, $\alpha=1,\dots,M$ in (38) are linear in B_α and $\boldsymbol{F}$ only depends on the critical variables A_i, $i=1,\dots,n_c$, then the Grassmann manifold B defined by

$$B_\alpha=0 \qquad \alpha=1,\dots,M \tag{40}$$

is invariant under the dynamics defined by (38). Therefore for any initial condition belonging to B the system will remain on B forever and its dynamics will be solely described in terms of the critical variables (equation (19a)).

Equivalently we can say that for times $t \gg \sup_\alpha |\mathrm{Re}\ \gamma\alpha|^{-1}$ all the variables B_α will relax to zero and therefore the subsequent asymptotic dynamics will be described by the slow variables A_i through (19a). By replacing in (18) $B_\alpha = 0$, $\alpha = 1, \ldots, M$, and expressing the M variables $\hat{\theta}_\alpha = \langle \boldsymbol{\theta}, \boldsymbol{\psi}_\alpha \rangle_E$ as functions of the n_c variables $\tilde{\theta}_j = \langle \boldsymbol{\theta}, \boldsymbol{\phi}_j \rangle_E$ we arrive at the M equations:

$$\begin{aligned} \hat{\theta}_1 &= \hat{\theta}_1(\tilde{\theta}_1, \ldots, \tilde{\theta}_{n_c}) \\ &\vdots \\ \hat{\theta}_M &= \hat{\theta}_M(\tilde{\theta}_1, \ldots, \tilde{\theta}_{nc}) \end{aligned} \tag{41}$$

which describe the Grassmann version of the stable manifold (Elphick *et al* 1986), Guckenheimer and Holmes 1983), and can be interpreted by saying that asymptotically the rapid variables $\hat{\theta}_\alpha$ follow the dynamics of the slow variables $\tilde{\theta}_j$.

The above description is just the Grassmann version of the adiabatic elimination method for differential equations with commuting variables (Haken 1977).

Once the non-linear change of variables (18) and equations (19a) and (19b) have been completely determined, appropriate initial conditions $A_i(t_0)$, $B_\alpha(t_0)$ have to be added to (19a) and (19b) such that $\boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_0$ through (18). A subsequent integration of (19a) and (19b) and replacement of its solutions $A_i(t, t_0)$, $B_\alpha(t, t_0)$ in (18) leads to the solution of the original problem (equation (15)). As we will see in § 5, there are a number of cases in which equations (19a) and (19b) can be easily integrated (Elphick (1986) considered in detail the case $\mathbb{J} = \mathrm{diag}(0, \mathrm{i}\omega, -\mathrm{i}\omega)$) and therefore the method presented in § 3 can be regarded as an integration technique of equations of the type (15).

5. Some examples

We introduce the notation $z^{n_1} z^{n_2} \ldots z^{n_q} lz\omega_1 \ldots \omega_s$ (if, for example, $\omega_1 = \omega_2 = \omega$ we use the notation 2ω (resp ω^2) for the semisimple (resp non-semisimple) case) to refer to the critical situation characterised by the matrix $\mathbb{J}$ given in (17).

5.1. *No critical situation*

Since there are no critical modes the original system is equivalent through (18) to the linear system

$$\mathrm{d}B_\alpha/\mathrm{d}t = \gamma_\alpha B_\alpha \qquad \alpha = 1, \ldots, M. \tag{42}$$

5.2. *z instability*

$\mathbb{J}$ is the 1×1 null matrix and its associated Arnold-Jordan matrix is (μ). Since there is only one critical mode we easily obtain the normal form for A_1 (in the unfolded situation)

$$\mathrm{d}A_1/\mathrm{d}t = \mu A_1 \tag{43}$$

and the equation for the stable modes is the same as in (42).

5.3. *ω instability (Hopf bifurcation)*

Since now we have two critical modes, we cannot form with them a monomial of odd degree. Therefore, the normal form is (with an unfolding parameter)

$$\mathrm{d}A/\mathrm{d}t = (\mu + \mathrm{i}\omega)A \tag{44a}$$

$$\mathrm{d}A^*/\mathrm{d}t = (\mu - \mathrm{i}\omega)A^*. \tag{44b}$$

From (35) we obtain the equation

$$\mathrm{i}\omega\left(A\frac{\partial^l}{\partial A} - A^*\frac{\partial^l}{\partial A^*}\right)H_\alpha = 0 \tag{45}$$

with solution $H_\alpha = k_\alpha AA^*$. Therefore the equations for the B_α, $\alpha = 1, \ldots, M$, are

$$\mathrm{d}B_\alpha/\mathrm{d}t = B_\alpha(\gamma_\alpha + k_\alpha AA^*). \tag{46}$$

For $\mu = 0$ equations (44*a*) and (44*b*) represent an overdamped complex Grassmann harmonic oscillator. We recall that for commuting variables the Hopf normal form represents a strictly non-linear oscillator.

5.4. *z^3 instability*

From (32) and (35) we obtain

$$\begin{aligned} &\mathscr{A}^+F_1 = 0 \qquad &&\mathscr{A}^+F_2 = F_1 \\ &A^+F_3 = F_2 \qquad &&\mathscr{A}^+H_\alpha = 0 \end{aligned} \tag{47}$$

where

$$\mathscr{A}^+ = A_1\,\partial^l/\partial A_2 + A_2\,\partial^l/\partial A_3.$$

Since there are only three critical modes, the solutions of (47) are given by $F_1 = F_2 = F_3 = 0$ and $H_\alpha = k_\alpha A_1A_2$. Therefore (38) is (in the unfolded case)

$$\frac{\mathrm{d}^3A_1}{\mathrm{d}t^3} + \mu_1\frac{\mathrm{d}^2A_1}{\mathrm{d}t^2} + \mu_2\frac{\mathrm{d}A_1}{\mathrm{d}t} + \mu_3A_1 = 0 \tag{48a}$$

$$\frac{\mathrm{d}B_\alpha}{\mathrm{d}t} = B_\alpha\left(\gamma_\alpha + k_\alpha A_1\frac{\mathrm{d}A_1}{\mathrm{d}t}\right) \qquad \alpha = 1, \ldots, M. \tag{48b}$$

5.5. *z^4 instability*

In this case (32) and (35) give

$$\begin{aligned} &\mathscr{A}^+F_1 = 0 \qquad &&\mathscr{A}^+F_2 = F_1 \\ &\mathscr{A}^+F_3 = F_2 \qquad &&\mathscr{A}^+F_4 = F_3 \\ &\mathscr{A}^+H_\alpha = 0 \end{aligned} \tag{49}$$

where

$$\mathscr{A}^+ = A_1\frac{\partial^l}{\partial A_2} + A_2\frac{\partial^l}{\partial A_3} + A_3\frac{\partial^l}{\partial A_4}.$$

Using the fact that the most general non-linear odd element in (A_1, A_2, A_3, A_4) is of the form

$$\alpha_1 A_1 A_2 A_3 + \alpha_2 A_1 A_2 A_4 + \alpha_3 A_1 A_3 A_4 + \alpha_4 A_2 A_3 A_4 \tag{50}$$

we readily obtain

$$\begin{aligned} F_1 &= \alpha A_1 A_2 A_3 \\ F_2 &= \alpha A_1 A_2 A_4 \\ F_3 &= \alpha A_1 A_3 A_4 \\ F_4 &= \alpha A_2 A_3 A_4. \end{aligned} \tag{51}$$

Similarly, by writing the most general even element as a linear combination of even monomials in (A_1, A_2, A_3, A_4) and replacing it in the last equation of (49) we obtain as solutions

$$H_\alpha = \begin{cases} k_\alpha A_1 A_2 \\ k'_\alpha A_1 A_2 A_3 A_4. \end{cases} \tag{52}$$

Finally the normal form can be further simplified by adding to (51) the following element in $\operatorname{ran}(\mathscr{A} - \mathbb{J})$:

$$\begin{aligned} F'_1 &= -\alpha A_1 A_2 A_3 \\ F'_2 &= -\alpha A_1 A_2 A_3 \\ F'_3 &= -\alpha A_1 A_3 A_4 \\ F'_4 &= 3\alpha A_2 A_3 A_4. \end{aligned} \tag{53}$$

The definitive equations are (with unfolding parameters $\mu_1, \mu_2, \mu_3, \mu_4$)

$$\begin{aligned} &\frac{\mathrm{d}^4 A_1}{\mathrm{d}t^4} + \mu_1 \frac{\mathrm{d}^3 A_1}{\mathrm{d}t^3} + \mu_2 \frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} + \mu_3 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \mu_4 A_1 = \nu \frac{\mathrm{d}A_1}{\mathrm{d}t} \frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} \frac{\mathrm{d}^3 A_1}{\mathrm{d}t^3} \\ &\frac{\mathrm{d}B_\alpha}{\mathrm{d}t} = B_\alpha \left(\gamma_\alpha + \eta_\alpha A_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \rho_\alpha A_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} \frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} \frac{\mathrm{d}^3 A_1}{\mathrm{d}t^3} \frac{\mathrm{d}^4 A_1}{\mathrm{d}t^4} \right) \qquad \alpha = 1, \ldots, M. \end{aligned} \tag{54}$$

5.6. 2ω instability

We now have four critical anticommuting variables A_1, A_1^*, A_2, A_2^*. The Jordan matrix $\mathbb{J}$ is given by

$$\mathbb{J} = \begin{pmatrix} \mathrm{i}\omega & 0 & 0 & 0 \\ 0 & -\mathrm{i}\omega & 0 & 0 \\ 0 & 0 & \mathrm{i}\omega & 0 \\ 0 & 0 & 0 & -\mathrm{i}\omega \end{pmatrix} \tag{55}$$

and consequently $\mathrm{e}^{\mathbb{J}^+ \eta}$ generates two independent rotations by $\omega\eta$ in the planes (A_1, A_1^*) and (A_2, A_2^*). Therefore using the invariance properties of § 3 we conclude that the functions G_α are linear combinations of monomials of even degree invariant under the above rotations and under the permutation $A_1 \to A_2$:

$$G_\alpha = \delta_\alpha (A_2^* A_2 + A_1^* A_1) + \beta_\alpha (A_1^* A_1 A_2^* A_2) \qquad \alpha = 1, \ldots, M. \tag{56}$$

The equivariance corresponding to $F=(F_1, F_1^*, F_2, F_1^*)$ in (39) leads to

$$e^{i\omega\eta}F_1(e^{-i\omega\eta}A_1, e^{i\omega\eta}A_1^*, e^{-i\omega\eta}A_2, e^{i\omega\eta}A_2^*)=F_1(A_1, A_1^*, A_2, A_2^*) \qquad (57a)$$

$$e^{-i\omega\eta}F_1^*(e^{-i\omega\eta}A_1, e^{i\omega\eta}A_1^*, e^{-i\omega\eta}A_2, e^{i\omega\eta}A_2^*)=F_1^*(A_1, A_1^*, A_2, A_2^*) \qquad (57b)$$

and similar expressions for F_2, F_2^*.

From (57*a*) and (57*b*) we easily obtain that the most general odd elements satisfying the equivariance property are given by

$$F_1=i(g_1A_2^*A_2A_1+g_2A_1^*A_2A_1) \qquad (58a)$$

$$F_2=i(g_1A_1^*A_1A_2+g_2A_2^*A_1A_2) \qquad (58b)$$

where g_1, g_2 are constants.

From (56), (58*a*) and (58*b*) we arrive at the following equations for $(A_1, A_1^*, A_2^*, A_2^*, B_\alpha, \alpha=1,\ldots,M)$:

$$\frac{1}{i}\frac{dA_1}{dt}=\omega A_1+(g_1A_2^*+g_2A_1^*)A_2A_1 \qquad (59a)$$

$$\frac{1}{i}\frac{dA_2}{dt}=\omega A_2+(g_1A_1^*+g_2A_2^*)A_1A_2 \qquad (59b)$$

$$\frac{dB_\alpha}{dt}=B_\alpha(\gamma_\alpha+\delta_\alpha(A_1^*A_1+A_2^*A_2)+\beta_\alpha A_1^*A_1A_2^*A_2). \qquad (60)$$

The equations for A_1^*, A_2^* are obtained from (59*a*) and (59*b*) by taking the involution.

Let us suppose now that the system considered is invariant under the independent reflections $A_1\to -A_1$, $A_2\to -A_2$. This implies $g_2=0$. By choosing g_1 real and defining the new variables

$$\begin{aligned} \phi_1&=A_1+iA_2^* \qquad & \phi_1^*&=A_1^*-iA_2 \\ \phi_2&=A_1-iA_2^* \qquad & \phi_2^*&=A_1^*+iA_2 \end{aligned} \qquad (61)$$

equations (59*a*) and (59*b*) can be equivalently written as

$$\frac{1}{i}\frac{d\phi_1}{dt}=\omega\phi_2+\tfrac{1}{2}g_1\phi_2^*\phi_2\phi_1 \qquad (62a)$$

$$\frac{1}{i}\frac{d\phi_2}{dt}=\omega\phi_1+\tfrac{1}{2}g_1\phi_1^*\phi_1\phi_2 \qquad (62b)$$

which is nothing but the anticommuting Thirring model describing Grassmann solitons if we identify ω with a mass m and t with the variable $\xi=\lambda x-\lambda^{-1}t$, $\lambda\in\mathbb{R}$, defined in a system moving with the soliton with a velocity λ^{-1} in the laboratory frame (Morris 1978).

References

Arnold V I 1971 *Russian Math. Surveys* **26** 29
—— 1977 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer)
Berezin F A 1966 *The Method of Second Quantisation* (New York: Academic)
Berezin F A and Marinov N S 1977 *Ann. Phys., NY* **104** 336
Elphick C 1987 *J. Math. Phys.* **28** to be published

Elphick C, Tirapegui E, Coullet P, Brachet M E and Iooss G 1986 *Physica* D submitted
Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Berlin: Springer)
Gürses M and Oǵuz Ö 1985 *Phys. Lett.* **108A** 437
Haken H 1977 *Synergetics—An Introduction* (Berlin: Springer)
Kulish D and Nissimov E 1976 *JETP Lett.* **24** 247
Kupershmidt B A 1984 *Phys. Lett.* **102A** 213
Morris H C 1978 *J. Math. Phys.* **19** 85
Ohnuki Y and Kamefuchi S 1980 *J. Math. Phys.* **21** 601
Olshanetsky M A 1983 *Commun. Math. Phys.* **88** 63
Roy Chowdhury A and Roy S 1986 *J. Math. Phys.* **27** 2464
Wahlquist H D and Estabrook F G 1975 *J. Math. Phys.* **16** 1